Acta Sci. Math., 41 (1979), 161—172

Purely contractive analytic functions and characteristic functions of non-contractions

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1. Introduction

If T is a bounded operator on a Hilbert space $\mathfrak{H}$, then the *characteristic function* of T is the operator valued analytic function

$$\Theta_T(\lambda) = [-TJ_T + \lambda J_{T^*} Q_{T^*}(I-\lambda T^*)^{-1} J_T Q_T] \,|\, \mathfrak{D}_T,$$

where $J_T = \operatorname{sgn}(I-T^*T)$, $J_{T*} = \operatorname{sgn}(I-TT^*)$, $Q_T = |I-T^*T|^{1/2}$, $Q_{T*} = |I-TT^*|^{1/2}$, and $\mathfrak{D}_T = J_T\mathfrak{H}$. $\Theta_T(\lambda)$ is defined for those complex numbers λ for which $I-\lambda T^*$ is boundedly invertible, and takes values which are continuous operators from $\mathfrak{D}_T$ to the space $\mathfrak{D}_{T*} = J_{T*}\mathfrak{H}$.

The characteristic function of a contraction T appears in the work of Sz.-Nagy and Foiaş [13] as the Fourier representation of a projection in the space of the unitary dilation of T. From this representation is obtained a functional model for T in terms of Θ_T. If $\Theta(\lambda): \mathfrak{D} \to \mathfrak{D}_*$ is an operator valued analytic function, then a contraction T can be constructed (using the same type of functional model) such that $\Theta = \Theta_T$, if and only if Θ is purely contractive, i.e., if $\|\Theta(\lambda)a\| < \|a\|$ whenever $|\lambda| < 1$, $a \in \mathfrak{D}$, $a \neq 0$.

In this paper we also consider operator valued analytic functions $\Theta(\lambda): \mathfrak{D} \to \mathfrak{D}_*$, but $\mathfrak{D}$ and $\mathfrak{D}_*$ are *Krein spaces* rather than Hilbert spaces (see Sec. 2 below), and thus the inner product is not assumed to be positive definite. We show that $\Theta = \Theta_T$ for some bounded operator T if and only if Θ is purely contractive and

Received October 11, 1977, in revised form June 24, 1978.

The results in this paper first appeared in the author's doctoral thesis [12] at the University of Toronto, under the supervision of Ch. Davis. Additional work on the paper, done at the University of Georgia, Athens, Georgia and the University of Missouri, Rolla, Missouri, was partially supported by an NSF grant.

fundamentally reducible, with respect to the indefinite inner products of $\mathfrak{D}$ and $\mathfrak{D}_*$ (see Sec. 3).

There have been previous papers (Brodskiĭ [3], Brodskiĭ, Gohberg, and Kreĭn [4], [5], Clark [6], and Ball [1]) giving necessary and sufficient conditions for an operator valued analytic function to be a characteristic function, but the conditions lack the simplicity of those presented here. In each paper, it is required that a certain function have an analytic extension to the unit disk (cf. Sec. 7); this requirement is eliminated here. In [6] and [1] it is also shown that the positive definiteness of certain functions is necessary and sufficient for Θ to be a characteristic function. We use this kind of result from [1] to prove our theorem, by showing that if Θ is a purely contractive fundamentally reducible analytic function, taking values which are operators between Krein spaces, then the associated kernel matrix function is positive definite (Theorem 3).

2. Krein spaces

An *indefinite inner product space* is a complex vector space $\mathfrak{K}$ on which is defined an inner product $[.,.]$ that is not assumed to be positive, i.e., it is possible for $[x, x]$ to be negative for some $x\in\mathfrak{K}$. We call $\mathfrak{K}$ a *Krein space* if there is an operator J on $\mathfrak{K}$ such that $J^2=I$, $J=J^*$, (i.e., $[Jx, y]=[x, Jy]$), and the *J-inner product*

$$(x, y) = [Jx, y] \quad (x, y\in\mathfrak{K}) \tag{2.1}$$

makes $\mathfrak{K}$ a Hilbert space. Such an operator J is called a *fundamental symmetry*. (See [2, Chapter V].)

The spaces $\mathfrak{D}_T=J_T\mathfrak{H}$ and $D_{T*}=J_{T*}\mathfrak{H}$, considered in Sec. 1, are Krein spaces with the indefinite inner products

$$[x, y] = (J_T x, y) \quad (x, y\in\mathfrak{D}_T)$$

and

$$[x, y] = (J_{T^*} x, y) \quad (x, y\in\mathfrak{D}_{T^*}).$$

(Here $(.,.)$ denotes the inner product on the Hilbert space $\mathfrak{H}$.) Clearly, J_T and J_{T*} are fundamental symmetries on $\mathfrak{D}_T$ and $\mathfrak{D}_{T*}$, respectively. We will always consider $\mathfrak{D}_T$ and $\mathfrak{D}_{T*}$ as Krein spaces, rather than as subspaces of the Hilbert space $\mathfrak{H}$.

In Krein spaces, the emphasis is always on the indefinite inner product, with the *J-norm* $\|x\|_J=[Jx, x]^{1/2}$ serving mainly to define the topology (the *strong* topology). Accordingly, if A is a continuous operator between Krein spaces $\mathfrak{K}$ and $\mathfrak{K}'$, we use A^* to denote the adjoint of A with respect to the indefinite inner products. If J and J' are fundamental symmetries on $\mathfrak{K}$ and $\mathfrak{K}'$, respectively, then the adjoint of A with respect to the J- and J'-inner products (2.1) will be denoted by $A^{(*)}$ and is given by $A^{(*)}=JA^*J'$ (see [2, Sec. VI. 2]).

Different fundamental symmetries J on a Krein space $\mathfrak{K}$ define different J-norms, but the strong topologies obtained coincide (see [12, Sec. 1.4] and [2]). Thus we can talk about *the* strong topology on a Krein space.

We will be needing the following:

Lemma 1. *If J is a symmetry on a Krein space $\mathfrak{K}$ (i.e., $J^2=I$, $J=J^*$) such that the J-inner product $(x, y)=[Jx, y]$ is positive, then J is a fundamental symmetry, i.e., the J-inner product makes $\mathfrak{K}$ a Hilbert space.*

Proof. See [2, Corollary V.1.2].

3. Purely contractive analytic functions. The main theorem

An *operator valued analytic function* is a function θ which is defined and analytic in D, the open unit disk in the complex plane, and which takes values that are continuous operators from a Krein space $\mathfrak{D}$ to a Krein space $\mathfrak{D}_*$. Θ is said to be *purely contractive* if, for each $\lambda \in D$,

$$[\Theta(\lambda)a, \Theta(\lambda)a] < [a, a] \quad (a \in \mathfrak{D},\ a \neq 0) \tag{3.1}$$

and

$$[\Theta(\lambda)^* a_*, \Theta(\lambda)^* a_*] < [a_*, a_*] \quad (a_* \in \mathfrak{D}_*,\ a_* \neq 0). \tag{3.2}$$

Remarks. In Hilbert space, (3.2) is implied by (3.1), but this is not true in general. (See, for example [9, Sec. 3]. Note that the results in [9] concerning expansive operators apply to contractive operators, with the inner product $-[.\,,.]$.) Also, in Hilbert space, it is not necessary to assume that the inequalities (3.1) and (3.2) are strict inequalities, except at $\lambda=0$, since the maximum modulus principle can then be used to get strict inequality for all $\lambda \in D$ (cf. [13, Sec. V.2.2]).

An operator A on a Krein space $\mathfrak{K}$ is said to be *fundamentally reducible* if there is a fundamental symmetry on $\mathfrak{K}$ commuting with A [2, Sec. VIII.1]. Suppose Θ is an operator valued analytic function, and let $\Theta_0=\Theta(0)$. We call the function Θ *fundamentally reducible* if the operators $\Theta_0^* \Theta_0$ (on $\mathfrak{D}$) and $\Theta_0 \Theta_0^*$ (on $\mathfrak{D}_*$) are fundamentally reducible.

If $\mathfrak{D}$ and $\mathfrak{D}'$ are two Krein spaces, then an operator $\tau: \mathfrak{D} \to \mathfrak{D}'$ is said to be *unitary* if it is continuous and invertible, and if $[\tau x, \tau x]=[x, x]$ for all $x \in \mathfrak{D}$. Two operator valued analytic functions $\Theta(\lambda): \mathfrak{D} \to \mathfrak{D}_*$ and $\Theta'(\lambda): \mathfrak{D}' \to \mathfrak{D}'_*$ are said to *coincide* if there are unitary operators $\tau: \mathfrak{D} \to \mathfrak{D}'$ and $\tau_*: \mathfrak{D}_* \to \mathfrak{D}'_*$ such that $\Theta'(\lambda)=\tau_* \Theta(\lambda)\tau^{-1}$ for all $\lambda \in D$.

We can now state the main result of this paper.

Theorem 1. *Suppose Θ is an operator valued analytic function. For Θ to coincide with the characteristic function of a bounded operator on a Hilbert space, it is necessary and sufficient that Θ be purely contractive and fundamentally reducible.*

The condition that Θ be fundamentally reducible can not be omitted from Theorem 1, as P. Jonas of Berlin has constructed an example of a (constant) purely contractive analytic function which does not coincide with the characteristic function of any bounded operator on a Hilbert space. The author is indebted to J. Bognár and B. Sz.-Nagy for pointing out the existence of this example.

4. Proof of necessity in Theorem 1

We assume here that Θ is an operator valued analytic function coinciding with Θ_T, for some operator T. Thus $\Theta_T(\lambda)$ is defined and analytic in the open unit disk D, and takes values which are continuous operators between the Krein spaces $\mathfrak{D}_T$ and $\mathfrak{D}_{T^*}$. To prove that Θ is purely contractive and fundamentally reducible, it clearly suffices to show that Θ_T is purely contractive and fundamentally reducible.

Recalling the definitions of the indefinite inner products on $\mathfrak{D}_T$ and $\mathfrak{D}_{T^*}$, we obtain

$$\Theta_T(\lambda)^* = [-T^*J_{T^*}+\bar{\lambda}Q_T(I-\bar{\lambda}T)^{-1}Q_{T^*}]|\mathfrak{D}_{T^*}, \tag{4.1}$$

and it follows, similarly to [13], relation (VI.1.4) (cf. [11]), that for $\lambda\in D$

$$I-\Theta_T(\lambda)^*\Theta_T(\lambda) = (1-|\lambda|^2)Q_T(I-\bar{\lambda}T)^{-1}(I-\lambda T^*)^{-1}J_TQ_T$$

and

$$I-\Theta_T(\lambda)\Theta_T(\lambda)^* = (1-|\lambda|^2)J_{T^*}Q_{T^*}(I-\lambda T^*)^{-1}(I-\bar{\lambda}T)^{-1}Q_{T^*}.$$

Consequently, for $\lambda\in D$,

$$[(I-\Theta_T(\lambda)^*\Theta_T(\lambda))a, a] = (1-|\lambda|^2)\|(I-\lambda T^*)^{-1}J_TQ_Ta\|^2 > 0,$$

where $a\in\mathfrak{D}_T$, $a\neq 0$; and

$$[(I-\Theta_T(\lambda)\Theta_T(\lambda)^*)a_*, a_*] = (1-|\lambda|^2)\|(I-\bar{\lambda}T)^{-1}Q_{T^*}a_*\|^2 > 0,$$

where $a_*\in\mathfrak{D}_{T^*}$, $a_*\neq 0$. Therefore, Θ_T is purely contractive.

Since $\Theta_T(0)^*\Theta_T(0)=T^*T$, it follows that $\Theta_T(0)^*\Theta_T(0)$ commutes with the fundamental symmetry $J_T=\operatorname{sgn}(I-T^*T)$ on $\mathfrak{D}_T$. Likewise, $\Theta_T(0)\Theta_T(0)^*=TT^*$ commutes with the fundamental symmetry $J_{T^*}=\operatorname{sgn}(I-TT^*)$ on $\mathfrak{D}_{T^*}$. Consequently, Θ_T is fundamentally reducible.

The proof of sufficiency in Theorem 1 occupies the next four sections.

5. Fundamental symmetries on $\mathfrak{D}$ and $\mathfrak{D}_*$

Suppose Θ is a purely contractive, fundamentally reducible analytic function. Let J_0 be a fundamental symmetry on $\mathfrak{D}$ commuting with $\Theta_0^* \Theta_0$. Then the operator $I-\Theta_0^*\Theta_0$ is not only self-adjoint but also J_0-self-adjoint (i.e., self-adjoint with respect to the J_0-inner product). Thus we can define

$$J = \operatorname{sgn}(I-\Theta_0^* \Theta_0) \quad \text{and} \quad Q = |I-\Theta_0^* \Theta_0|^{1/2},$$

where these are computed using the J_0-self-adjoint functional calculus on the Hilbert space $\mathfrak{D}$ with the J_0-inner product. J and Q commute with J_0 and hence are self-adjoint (as well as J_0-self-adjoint).

Since Θ is assumed to be purely contractive, $I-\Theta_0^*\Theta_0$ is injective. Therefore we conclude that $J^2=I$ and Q has range dense in $\mathfrak{D}$. Also, we have

$$[JQh, Qh] = [(I-\Theta_0^* \Theta_0)h, h] \geqq 0 \quad (h \in \mathfrak{D}),$$

and consequently $[Jh, h] \geqq 0$ for all $h \in \mathfrak{D}$. It follows from Lemma 1 that J is a fundamental symmetry on $\mathfrak{D}$.

Similarly, if J_{0*} is a fundamental symmetry on $\mathfrak{D}_*$ commuting with $\Theta_0\Theta_0^*$, and if we define

$$J_* = \operatorname{sgn}(I-\Theta_0\Theta_0^*),$$

using the J_{0*}-self-adjoint functional calculus on $\mathfrak{D}_*$, then J_* is a fundamental symmetry on $\mathfrak{D}_*$.

The operators J and J_* do not depend on the choice of fundamental symmetries J_0 and J_{0*}. Indeed, suppose J_1 is another fundamental symmetry on $\mathfrak{D}$ commuting with $\Theta_0^*\Theta_0$, and let $J'=\operatorname{sgn}(I-\Theta_0^*\Theta_0)$ and $Q'=|I-\Theta_0^*\Theta_0|^{1/2}$ be computed with respect to the J_1-inner product. Then J' and Q' are also J_0-self-adjoint, and $JQ^2=J'Q'^2$. Since both J and J' are J_0-unitary, and both Q^2 and Q'^2 are J_0-positive, the uniqueness of the J_0-polar decomposition of JQ^2 implies $J=J'$. The argument for J_* is similar.

Proposition 1. *Θ_0^* is the same as $\Theta_0^{(*)}$, the adjoint of Θ_0 with respect to the J- and J_*-inner products on $\mathfrak{D}$ and $\mathfrak{D}_*$.*

Proof. $\Theta_0^{(*)}=J\Theta_0^*J_*=J(J\Theta_0^*)=\Theta_0^*$ (cf. [7, Sec. 2]).

Although Clark [6] and Ball [1] do not approach the subject of fundamental symmetries on $\mathfrak{D}$ and $\mathfrak{D}_*$ in the same manner as we have done here, the end result is the same. Their approach is to begin with $\mathfrak{D}$ and $\mathfrak{D}_*$ as Hilbert spaces and subsequently impose on them the Krein space structures derived from the symmetries $J=\operatorname{sgn}(I-\Theta_0^*\Theta_0)$ and $J_*=\operatorname{sgn}(I-\Theta_0\Theta_0^*)$. Thus it is implicit in the definitions of the inner products in [6] and [1] that Θ is to be taken to be fundamentally reducible.

BRODSKIĬ [3], and BRODSKIĬ, GOHBERG, and KREĬN [4], [5] consider $\mathfrak{D}$ and $\mathfrak{D}_*$ as Hilbert spaces, but do not, however, assume $J=\operatorname{sgn}(I-\Theta_0^*\Theta_0)$. Instead, they deal with a more general situation in which the object studied is not a single operator T but an *uzel*, a collection of operators and Hilbert spaces. A $\mathcal{Y}$-uzel is a collection of spaces $\mathfrak{H}$, $\mathfrak{G}$ and operators T, R, J, where $T\colon \mathfrak{H}\to\mathfrak{H}$, $R\colon \mathfrak{G}\to\mathfrak{H}$, J is a symmetry on $\mathfrak{G}$, and $I-T^*T=RJR^*$. The particular case of interest to us is when $\mathfrak{G}=\mathfrak{D}_T$, $R=Q_T$, and $J=J_T$.

6. The theorem of Ball

We wish to apply [1, Theorem 2], but some differences in notation need to be cleared up first. Let us define $\bar{\Theta}(\lambda)=\Theta(\bar{\lambda})^*$. In [1], the characteristic function B_T is $B_T=\bar{\Theta}_T$ (cf. (4.1)), and so the condition given in [1, Theorem 2] for B to be a characteristic function must be written with $B=\bar{\Theta}$. Also, operators in [1] are assumed to act between Hilbert spaces, whereas here we consider them as acting between Krein spaces. The concept of adjoint must therefore be interpreted appropriately. Proposition 1 shows that the definitions of J and J_* given here are the same as those in [1] (with $B=\bar{\Theta}$).

It should be noted that in [1, Theorem 2] it is being asserted that $\tau B=B_T\tau_*$ for some operator T, where τ and τ_* are unitary operators between Hilbert spaces. Since $J=\operatorname{sgn}(I-B(0)B(0)^*)$ and $J_T=\operatorname{sgn}(I-B_T(0)B_T(0)^*)$, it follows that $\tau J=J_T\tau$. Similarly, $\tau_*J_*=J_{T*}\tau_*$, and we deduce that τ and τ_* are also unitary operators between Krein spaces. Thus we have coincidence of B and B_T in the sense of Sec. 3.

We can now state Ball's theorem, using our notation:

Theorem 2. ([1, Theorem 2]) *Let $\Theta(\lambda)\colon \mathfrak{D}\to\mathfrak{D}_*$ be an operator valued analytic function. Then θ coincides with the characteristic function of some bounded operator on a Hilbert space if and only if*

(i) *Θ is fundamentally reducible,*

(ii) *$I-\Theta_0^*\Theta_0$ and $I-\Theta_0\Theta_0^*$ are injective, and*

(iii) *the operator matrix*

$$k(\mu,\lambda)=\begin{pmatrix}(1-\lambda\bar{\mu})^{-1}(I-\Theta(\mu)^*\Theta(\lambda)) & (\lambda-\bar{\mu})^{-1}(\Theta(\bar{\lambda})^*-\Theta(\mu)^*)\\ (\lambda-\bar{\mu})^{-1}(\Theta(\lambda)-\Theta(\bar{\mu})) & (1-\lambda\bar{\mu})^{-1}(I-\Theta(\bar{\mu})\Theta(\bar{\lambda})^*)\end{pmatrix} \tag{6.1}$$

is positive definite on some neighborhood $\mathcal{U}$ of zero, i.e.,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}[k(\mu_j,\mu_i)(f_i\oplus g_i),(f_j\oplus g_j)]\geqq 0 \tag{6.2}$$

for all $n\geqq 1$ and for all $f_i\in\mathfrak{D}$, $g_i\in\mathfrak{D}_*$, and $\mu_i\in\mathcal{U}$ $(i=1,2,\dots,n)$.

Note. $k(\mu, \lambda)$ is considered as an operator on the Krein space $\mathfrak{D}\oplus\mathfrak{D}_*$. The matrix in [1] is obtained by setting $B=\bar{\Theta}$ and considering $(I\oplus J_*)k(\bar{z}, \bar{w})(J\oplus I)$. The neighborhood D considered in [1] is denoted by $\mathscr{U}$ in the above theorem. (In this paper, D denotes the open unit disk.)

7. The functions Φ and Ω

In view of Theorem 2, it remains to show that (6.2) is valid whenever Θ is purely contractive and fundamentally reducible.

If we consider, for the moment, $\mathfrak{D}$ and $\mathfrak{D}_*$ as Hilbert spaces, with the J- and J_*-inner products, then Proposition 1 shows that Θ_0 has a polar decomposition

$$\Theta_0 = U(\Theta_0^*\Theta_0)^{1/2} = (\Theta_0\Theta_0^*)^{1/2}U. \tag{7.1}$$

(This decomposition can also be done in Krein space. See [10].) Following the argument used in [1, Sec. 1.3 and Sec. 2.1], we can assume that the codimension of $(\Theta_0^*\Theta_0)^{1/2}\mathfrak{D}$, in the Hilbert space $\mathfrak{D}$, equals the codimension of $(\Theta_0\Theta_0^*)^{1/2}\mathfrak{D}_*$, in the Hilbert space $\mathfrak{D}_*$. Then U can be chosen to be a unitary operator between the Hilbert spaces $\mathfrak{D}$ and $\mathfrak{D}_*$. From (7.1) we have $U(\Theta_0^*\Theta_0)=(\Theta_0\Theta_0^*)U$, and hence $UJ=J_*U$. Thus U is also a unitary operator between the Krein spaces $\mathfrak{D}$ and $\mathfrak{D}_*$.

If the vector $f\in\mathfrak{D}$ satisfies $(I+U^*\Theta(\lambda))f=0$, for some $\lambda\in D$, then it follows that

$$[f, f] = [U^*\Theta(\lambda)f, U^*\Theta(\lambda)f] = [\Theta(\lambda)f, \Theta(\lambda)f].$$

Hence, since Θ is purely contractive, $f=0$ and we conclude that $I+U^*\Theta(\lambda)$ is injective for all $\lambda\in D$. Similarly, $I+\Theta(\lambda)^*U$ is injective and thus $I+U^*\Theta(\lambda)$ has range dense in $\mathfrak{D}$, for all $\lambda\in D$. Hence, for each $\lambda\in D$, we can define

$$\Phi(\lambda) = (I-U^*\Theta(\lambda))(I+U^*\Theta(\lambda))^{-1}J, \tag{7.2}$$

an operator with domain dense in $\mathfrak{D}$.

Note that the operator $I+U^*\Theta_0=I+(\Theta_0^*\Theta_0)^{1/2}$ is boundedly invertible, and hence $(I+U^*\Theta(\lambda))^{-1}$ is analytic for all λ belonging to some neighborhood $\mathscr{U}$ of zero, with $\mathscr{U}\subset D$ (cf. [8, Sec. VII.1.1]). Consequently, $\Phi(\lambda)$ is analytic for $\lambda\in\mathscr{U}$. Clearly, we can assume that $\mathscr{U}$ is closed under complex conjugation.

Remark. In [3], [4], [5], [6], and [1] assumptions are made which amount to assuming that $\Phi(\lambda)$ $(\lambda\in\mathscr{U})$ extends to an analytic function on D. However, it is not necessary to make this assumption here.

We obtain from (7.2) that, for each $\lambda\in D$,

$$\Phi(\lambda)J = 2(I+U^*\Theta(\lambda))^{-1}-I, \tag{7.3}$$

and thus $\Phi(\lambda)J$ is closed. Consequently, for each $\lambda \in D$, $\Phi(\lambda)$ is closed. Also, (7.3) implies that

$$\Phi(\lambda)^{(*)}J = 2(I+\Theta(\lambda)^* U)^{-1} - I$$

(recall the notation introduced in Sec. 2), and hence

$$\Phi(\lambda)^{(*)} = (I-\Theta(\lambda)^* U)(I+\Theta(\lambda)^* U)^{-1}J. \tag{7.4}$$

For $\lambda \in D$, an arbitrary vector in the domain of $\Phi(\lambda)$ is of the form $f=J(I+U^*\Theta(\lambda))g$, where $g \in \mathfrak{D}$. If $(.\,,.)$ denotes the J-inner product on $\mathfrak{D}$, then we can readily deduce that

$$\operatorname{Re}(\Phi(\lambda)f, f) = [g, g] - [\Theta(\lambda)g, \Theta(\lambda)g] \geqq 0. \tag{7.5}$$

It follows that

$$\|(I+\Phi(\lambda))f\|_J^2 \geqq \|f\|_J^2 + \|\Phi(\lambda)f\|_J^2 \geqq \|(I-\Phi(\lambda))f\|_J^2, \tag{7.6}$$

and hence $I+\Phi(\lambda)$ is injective with closed range.

By a similar argument, we deduce from (7.4) that $I+\Phi(\lambda)^{(*)}$ is injective, and thus $I+\Phi(\lambda)$ has dense range. Therefore, the operator $I+\Phi(\lambda)$ is bijective and closed, and consequently (by the closed graph theorem) boundedly invertible (for all $\lambda \in D$).

We can now make the definition

$$\Omega(\lambda) = (I-\Phi(\lambda))(I+\Phi(\lambda))^{-1} \quad (\lambda \in D).$$

By the preceding paragraph, $\Omega(\lambda)$ has domain equal to $\mathfrak{D}$ and, by (7.6), $\Omega(\lambda)$ is a contraction on the Hilbert space $\mathfrak{D}$ with the J-inner product. Since Θ is purely contractive, equality holds in (7.5) only if $f=0$, and hence the same is true in (7.6). It then readily follows that Ω is purely contractive.

$\Phi(\lambda)$ is known to be analytic only in a neighborhood of zero, and thus the analyticity of $\Omega(\lambda)$ in D is not immediately obvious. We can write

$$I+\Phi(\lambda) = \Psi(\lambda)(I+U^*\Theta(\lambda))^{-1}J, \tag{7.7}$$

where $\Psi(\lambda)=J(I+U^*\Theta(\lambda))+(I-U^*\Theta(\lambda))$. Since $I+\Phi(\lambda)$ is boundedly invertible, (7.7) implies that $\Psi(\lambda)$ is boundedly invertible. Since $\Psi(\lambda)$ is analytic in D, so is $\Psi(\lambda)^{-1}$ (see [8, Sec. VII.1.1]), and therefore the function

$$(I+\Phi(\lambda))^{-1} = J(I+U^*\Theta(\lambda))\Psi(\lambda)^{-1}$$

is analytic in D. Finally, we note that

$$\Omega(\lambda) = 2(I+\Phi(\lambda))^{-1} - I \quad (\lambda \in D),$$

and thus $\Omega(\lambda)$ is analytic in D.

We have now shown that Ω is a purely contractive analytic function on the Hilbert space $\mathfrak{D}$ (with the J-inner product) and thus (by [13, Theorem VI.3.1]) Ω coincides with the characteristic function of some contraction S, i.e., $\tau_* \Omega(\lambda) =$

$=\Theta_S(\lambda)\tau$ for some unitary operators $\tau:\mathfrak{D}\to\mathfrak{D}_S$ and $\tau_*:\mathfrak{D}\to\mathfrak{D}_{S^*}$ ($\mathfrak{D}$ considered as a Hilbert space). It then follows that, for each $n\geqq 1$, and for $a_i, b_i\in\mathfrak{D}$, $\mu_i\in D$ $(i=1,2,\ldots,n)$ we have (using the J-inner product)

(7.8)

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\{((1-\mu_i\bar\mu_j)^{-1}(I-\Omega(\mu_j)^{(*)}\Omega(\mu_i))a_i, a_j)+((\mu_i-\bar\mu_j)^{-1}(\Omega(\mu_i)-\Omega(\bar\mu_j))a_i, b_j)+$$
$$+((\mu_i-\bar\mu_j)^{-1}(\Omega(\bar\mu_i)^{(*)}-\Omega(\mu_j)^{(*)})b_i, a_j)+((1-\mu_i\bar\mu_j)^{-1}(I-\Omega(\bar\mu_j)\Omega(\bar\mu_i)^{(*)})b_i, b_j)\}=$$
$$=\left\|\sum_{i=1}^{n}\{(I-\mu_iS^*)^{-1}Q_S\tau a_i+(I-\mu_iS)^{-1}Q_{S^*}\tau_*b_i\}\right\|^2\geqq 0$$

(cf. [1, Sec. 1.5]).

8. Positive definiteness of $k(\mu,\lambda)$

We now prove a series of results that will enable us to write (6.2) in the form (7.8) and thus establish the positive definiteness of $k(\mu,\lambda)$.

Proposition 2. *For $\lambda\in D$, $I+\Phi(\lambda)J$ is boundedly invertible and $U^*\Theta(\lambda)=$ $=(I-\Phi(\lambda)J)(I+\Phi(\lambda)J)^{-1}$.*

Proof. The equation $I+\Phi(\lambda)J=2(I+U^*\Theta(\lambda))^{-1}$, obtained from (7.3), shows that $I+\Phi(\lambda)J$ is boundedly invertible and that

$$U^*\Theta(\lambda)(I+\Phi(\lambda)J)=[(I+U^*\Theta(\lambda))-(I-U^*\Theta(\lambda))](I+U^*\Theta(\lambda))^{-1}=I-\Phi(\lambda)J.$$

Lemma 2. *Suppose that A_1 and A_2 are bounded operators with $I+A_i$ boundedly invertible $(i=1,2)$. Then if $B_i=(I-A_i)(I+A_i)^{-1}$ $(i=1,2)$, we have*

$$I-B_1B_2=2(I+A_1)^{-1}(A_1+A_2)(I+A_2)^{-1}$$

and

$$B_2-B_1=2(I+A_1)^{-1}(A_1-A_2)(I+A_2)^{-1}.$$

Proof. We simply perform the calculations:

$$I-B_1B_2=(I+A_1)^{-1}[(I+A_1)(I+A_2)-(I-A_1)(I-A_2)](I+A_2)^{-1}=$$
$$=2(I+A_1)^{-1}(A_1+A_2)(I+A_2)^{-1};$$
$$B_2-B_1=(I+A_1)^{-1}[(I+A_1)(I-A_2)-(I-A_1)(I+A_2)](I+A_2)^{-1}=$$
$$=2(I+A_1)^{-1}(A_1-A_2)(I+A_2)^{-1}.$$

In the following, $\mathscr{U}$ is the neighborhood of zero where $\Phi(\lambda)$ is analytic (Sec. 7).

Lemma 3. *For $\lambda, \mu \in \mathscr{U}$ we have*

(i) $I-\Theta(\mu)^*\Theta(\lambda) =$

$$= \big(I+J\Phi(\mu)^*\big)^{-1}\big(I+\Phi(\mu)^{(*)}\big)[I-\Omega(\mu)^{(*)}\Omega(\lambda)]\big(I+\Phi(\lambda)\big)J\big(I+\Phi(\lambda)J\big)^{-1},$$

(ii) $\Theta(\lambda)-\Theta(\bar{\mu}) =$

$$= U\big(I+\Phi(\bar{\mu})J\big)^{-1}\big(I+\Phi(\bar{\mu})\big)[\Omega(\lambda)-\Omega(\bar{\mu})]\big(I+\Phi(\lambda)\big)J\big(I+\Phi(\lambda)J\big)^{-1},$$

(iii) $\Theta(\bar{\lambda})^*-\Theta(\mu)^* =$

$$= \big(I+J\Phi(\mu)^*\big)^{-1}\big(I+\Phi(\mu)^{(*)}\big)[\Omega(\bar{\lambda})^{(*)}-\Omega(\mu)^{(*)}]\big(I+\Phi(\bar{\lambda})^{(*)}\big)J\big(I+\Phi(\bar{\lambda})^*\big)^{-1}U^*,$$

(iv) $I-\Theta(\bar{\mu})\Theta(\bar{\lambda})^* =$

$$= U\big(I+\Phi(\bar{\mu})J\big)^{-1}\big(I+\Phi(\bar{\mu})\big)[I-\Omega(\bar{\mu})\Omega(\bar{\lambda})^{(*)}]\big(I+\Phi(\bar{\lambda})^{(*)}\big)J\big(I+J\Phi(\bar{\lambda})^*\big)^{-1}U^*.$$

Proof. Since $\mathscr{U}$ is assumed to be self-conjugate, both $\Phi(\lambda)$ and $\Phi(\bar{\lambda})$ are bounded for $\lambda \in \mathscr{U}$. It follows from Sec. 7 and Proposition 2 that all operators appearing above are bounded.

We know by Proposition 2 that

$$U^*\Theta(\lambda) = \big(I-\Phi(\lambda)J\big)\big(I+\Phi(\lambda)J\big)^{-1},$$

and we obtain from this (and the adjoint relation), by means of Lemma 2, the equations

$$I-\Theta(\mu)^*\Theta(\lambda) = 2\big(I+J\Phi(\mu)^*\big)^{-1}\big(J\Phi(\mu)^*+\Phi(\lambda)J\big)\big(I+\Phi(\lambda)J\big)^{-1} =$$

$$= 2\big(I+J\Phi(\mu)^*\big)^{-1}\big(\Phi(\mu)^{(*)}+\Phi(\lambda)\big)J\big(I+\Phi(\lambda)J\big)^{-1}.$$

We also have $\Omega(\lambda)=\big(I-\Phi(\lambda)\big)\big(I+\Phi(\lambda)\big)^{-1}$ and hence, using Lemma 2 again, we obtain

$$I-\Omega(\mu)^{(*)}\Omega(\lambda) = 2\big(I+\Phi(\mu)^{(*)}\big)^{-1}\big(\Phi(\mu)^{(*)}+\Phi(\lambda)\big)\big(I+\Phi(\lambda)\big)^{-1}.$$

Combining these two results gives (i).

Equations (ii), (iii), and (iv) are proved similarly.

Theorem 3. *$k(\mu, \lambda)$ is positive definite on a neighborhood $\mathscr{U}$ of zero.*

Proof. Let $\mathcal{U}$ be the neighborhood of zero on which $\Phi(\lambda)$ is analytic. Then for each $n \geqq 1$, and for $f_i \in \mathfrak{D}$, $g_i \in \mathfrak{D}_*$, and $\mu_i \in \mathcal{U}$ $(i=1, 2, \dots, n)$ we have (from (6.1))

$$\text{(8.1)} \qquad \sum_{i=1}^{n} \sum_{j=1}^{n} [k(\mu_j, \mu_i)(f_i \oplus g_i), (f_j \oplus g_j)] =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \{[(1-\mu_i\bar{\mu}_j)^{-1}(I-\Theta(\mu_j)^*\Theta(\mu_i))f_i, f_j] + [(\mu_i-\bar{\mu}_j)^{-1}(\Theta(\mu_i)-\Theta(\bar{\mu}_j))f_i, g_j] +$$

$$+ [(\mu_i-\bar{\mu}_j)^{-1}(\Theta(\bar{\mu}_i)^*-\Theta(\mu_j)^*)g_i, f_j] + [(1-\mu_i\bar{\mu}_j)^{-1}(I-\Theta(\bar{\mu}_j)\Theta(\bar{\mu}_i)^*)g_i, g_j]\}.$$

By Lemma 3,

$$\text{(8.2)} \qquad [(I-\Theta(\mu_j)^*\Theta(\mu_i))f_i, f_j] =$$

$$= [(I+J\Phi(\mu_j)^*)^{-1}(I+\Phi(\mu_j)^{(*)})(I-\Omega(\mu_j)^{(*)}\Omega(\mu_i))(I+\Phi(\mu_i))J(I+\Phi(\mu_i)J)^{-1}f_i, f_j] =$$

$$= [(I+\Phi(\mu_j)^{(*)})(I-\Omega(\mu_j)^{(*)}\Omega(\mu_i))(I+\Phi(\mu_i))J(I+\Phi(\mu_i)J)^{-1}f_i, (I+\Phi(\mu_j)J)^{-1}f_j] =$$

$$= (J(I+\Phi(\mu_j)^{(*)})(I-\Omega(\mu_j)^{(*)}\Omega(\mu_i))(I+\Phi(\mu_i))J(I+\Phi(\mu_i)J)^{-1}f_i, (I+\Phi(\mu_j)J)^{-1}f_j) =$$

$$= ((I-\Omega(\mu_j)^{(*)}\Omega(\mu_i))a_i, a_j),$$

where $a_i = (I+\Phi(\mu_i))J(I+\Phi(\mu_i)J)^{-1}f_i \quad (i=1, 2, \dots, n)$.

We also make the definition

$$b_i = (I+\Phi(\bar{\mu}_i)^{(*)})J(I+J\Phi(\bar{\mu}_i)^*)^{-1}U^*g_i \quad (i = 1, 2, \dots, n).$$

Then, by applying Lemma 3 to the terms of (8.1) in the same manner as (8.2) above, we deduce that (8.1) is the same as (7.8). Therefore, $k(\mu, \lambda)$ is positive definite on $\mathcal{U}$.

Theorem 3, in conjunction with Theorem 2, completes the proof of Theorem 1.

9. Conclusion

Theorem 1 establishes for certain non-contractions a result which generalizes a result of Sz.-Nagy and Foiaş for contractions. The proof, however, does not make use of a Sz.-Nagy and Foiaş type construction of the operator model, but instead relies on the much less geometric model obtained by Ball. When Θ is bounded, i.e., $\sup_{\lambda \in D} \|\Theta(\lambda)\| < \infty$, a model can be obtained that closely resembles the Sz.-Nagy and Foiaş model (see [12]), and this will be the subject of a future paper.

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